Fractals and Noise – Creation and Application

Short Tutorial

Fractal Geometry Properties and Exploitation: 30+ years of Images

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Contents of Presentation

- Introduction to Fractal Geometry.
- Properties – Deterministic Fractals.
- Noise – Creation and Images.
- Language for Fractals.
- Extra bits:
  - Fractal Compression
  - Fractal Segmentation

Introduction to Fractals and Chaos

- Algorithms used to generate fractals and chaotic fields depend on:
  understanding a physical (e.g. non-linear) system;
  clear and concise (mathematical) definition(s) of the field properties.

Resources at:
http://wiki.rcs.manchester.ac.uk/community/Fractal_Resources_Tutorial

Research Computing
The University of Manchester
Web: http://www.rcs.manchester.ac.uk/
Example Applications

- Signal Processing:
  Time Series Analysis, Speech Recognition
- Image Processing:
  Fractal Compression, Fractal Dimension Segmentation
- Simulation:
  Terrain Modelling, Image Synthesis, Music, Stochastic Fields

Example Applications

- Financial:
  Fractal Market Analysis, Futures Markets
- Medicine:
  Histology, Monitoring, Epidemiology
- Military:
  Visual Camouflage, Covert Digital Communications

Brief History of Fractals/Chaos

- 1815-1897: K W T Weierstrass
  Nowhere Differentiable Functions
- 1854-1912: J H Poincare
  Non-Deterministic (Chaotic) Dynamics
- 1900s:
  Julia, Koch & others
  Julia sets (simple fractals) - arise in connection with the iteration of a function of a complex variable

Brief History of Fractals/Chaos

- 1886-1971: P Levy
  Fractal Random Walks (Random Fractals)
- 1960s: E Lorenz
  Nonlinear Systems Dynamics and Chaos
- 1970s: B Mandelbrot
  Mandelbrot sets and the development of a general theory on the ‘Fractal Geometry of Nature’ (1975)
Brief History of Fractals/Chaos

1815-1897: K.W.T. Weierstrass
- Nowhere Differentiable Functions

\[ F(x) = |x| \]

Matlab – speed.

Aside: Fourier Series

\[ f(x) = \sum_{n=1}^{\infty} b^n \cos(\alpha_n x) \]

1815-1897: K.W.T. Weierstrass
- Dreadful Plague,
- Hermite

Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show to the arguments of our fathers, and they will never have any other use.

- Poincaré

\[ \text{step}(x) = \sum_{i=0}^{\infty} \frac{\sin((i + 2 + 1)x)}{i + 2 + 1} \]
Hilbert – one of the first useful fractals

- Hilbert Space Filling Curve

Suddenly we have uses:
- Dimension reduction
- Dithering style
- Compression codecs

Properties: Function $f$, is a one-to-one mapping, is onto, and continuous function.

There is no continuous inverse mapping.

Robert Brown – Non-Differentiation in nature

varies in the wildest way in magnitude and direction, and does not tend to a limit as the time taken for an observation decrease.

nature contains suggestions of non-differentiable as well as differentiable processes.

Jean Perrin

Step: 32, Length: 6392

Robert Brown – non-differentiation in nature

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Jean Perrin

Step: 16, Length: 8747
Robert Brown – non-differentiation in nature

does not tend to a limit as the time taken for an observation decrease

nature contains suggestions of non-differentiable as well as differentiable processes

Jean Perrin

Texture and Fractals

Texture is an elusive notion which mathematicians and scientists find hard to grasp

Much of Fractal Geometry can be considered to be an intrinsic study of texture

Benoit Mandelbrot

The Fractal Geometry of Nature: Clouds

Photo from Manchester, UK
Islamic Art: Self-Repeating Patterns

Self-Similarity by M C Escher

Self-Similarity by K Hokusai - Japanese Art from the 1800s

Self-Similarity by Snow Photos by Patricia Rasmussen
This movement depicts the island as a physical object. There are three structural principles at work: 1) reading the silhouette of the island seen from the shore (east to west) as a graph with range/density along the y-axis and time (19') along the x-axis 2) alternating "inbreaths" and "outbreaths" of varying durations calculated by careful measurement of the inlets and outlets of the island's shoreline, as shown on a hand-drawn map 3) a series of soundscapes representing the various environments encountered during a brisk walk around the island (which takes 19').

- The word Fractal was introduced by B. Mandelbrot in the 1970s.
  
  The term fractal is derived from the Latin adjective fractus. The corresponding Latin verb frangere means ‘to break’, to create irregular fragments. In addition to ‘fragmented’ fractus should also mean ‘irregular’, both meanings being preserved in fragment.
Introduction of CG in the 70's

“Objets Fractals” – published 1975

“Fractals Form Chance and Dimension” – published 1977

“Zooming into the Mandelbrot Set”

Properties of Fractals: The Von Koch Curve

Initiator:

Production Rule:

Repeating the Production rule again and again and again.
Properties of Fractals: The Von Koch Island

Joining three Curves gives us an Island.

Fundamental properties of a fractal signal

1. It is nowhere differentiable.
2. It is infinitely long.
3. It is finitely bounded.
4. It has a notion of self-similarity at each level.

Dimension of a Fractal

The dimension is the rate at which the signal approaches infinite length. A smooth signal has a lower dimension.

\[ N_m^D = 1, \text{ where } D_s = -\frac{\ln N}{\ln r} \]

\[ D = \frac{\log 4}{\log 3} \approx 1.262 \]

<table>
<thead>
<tr>
<th>Fractal Dimension</th>
<th>Fractal Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 &lt; D &lt; 1</td>
<td>Fractal Dust</td>
</tr>
<tr>
<td>1 &lt; D &lt; 2</td>
<td>Fractal Signal</td>
</tr>
<tr>
<td>2 &lt; D &lt; 3</td>
<td>Fractal Surface or Image</td>
</tr>
<tr>
<td>3 &lt; D &lt; 4</td>
<td>Fractal Volume</td>
</tr>
</tbody>
</table>
Dimension of a Fractal

\[ \theta = 72^\circ, D_\theta = 1.448 \]
\[ \theta = 60^\circ, D_\theta = 1.262 \]
\[ \theta = 54^\circ, D_\theta = 1.177 \]
\[ \theta = 45^\circ, D_\theta = 1.129 \]
\[ \theta = 18^\circ, D_\theta = 1.018 \]
\[ \theta = 12^\circ, D_\theta = 1.008 \]

Box Counting – Dimension

\[ D_B = \lim_{\delta \to 0} -\frac{\ln N(\delta)}{\ln \delta} \]

Fractional Brownian Motion

fBm has zero mean Gaussian distribution increments with variance:
\[ \langle [B_H(t + x) - B_H(t)]^2 \rangle \propto |x|^{2H} \]

with scaling property
\[ \langle \Delta B_H(rx)^2 \rangle \propto r^H \langle \Delta B_H(x)^2 \rangle \]

Most famous case when, \( H = 1/2 \)

Fractional Brownian Motion

fBm is related to Box Counting
\[ \Delta B_H = \Delta t^H = \frac{1}{N^{1-H}} = N^{-H} \quad N = 1/(\Delta t) \]

\[ \Delta B_H \Delta t = \frac{N^{-H}}{N^{-1}} = N^{1-H} \]

So the number of boxes,
\[ N(\delta) = NN^{1-H} = N^{2-H} \quad D = 2 - H \]
Power Spectrum Method

We consider this a ‘favoured’ system;

We wish to match a power spectrum decay rate, that has similar properties to fBm.

Euclidean Objects in Fourier Space

Fractal Objects in Fourier Space

Frequency Spectrum and Log-Log

- Frequency spectrum $F$ is proportional to $(\text{Frequency})^q$ or $F = c/k^q$
  
  where $c$ is a constant and $k$ denotes frequency.
- Take logs of both side and we get $\log(F) = C - q\log(k)$ where $C = \log(c)$
- Equation describes a straight line with a negative gradient determined by value of $q$. 
Fractal Signal of Koch Curve and its Log-Log Frequency Spectrum

The Fractal Dimension D and q (the Fourier Dimension)

- The larger the value of the fractal dimension D the smaller the value of q.
- D and q must be related in some way.
- For fractal functions: $q = (5 - 2D)/2; 1 < D < 2$.
- The Fourier Dimension determines the ‘roughness’, ‘texture’ or frequency characteristics or spectrum of the fractal.

Random Fractals and Nature: Stochastic Noise

- Very few natural shapes are regular fractals.
- The majority of natural shapes are statistically self-affine fractals.
- As we zoom into a random fractal, the shape changes, but the distribution of lengths (texture) remains the same.

Fundamental Definition of a Random Fractal Signal: Frequency Spectrum

- Let f be a random function with spectrum F.
- Let n be ‘white’ noise with spectrum N.
- If $F = N/k^q$ where k is frequency, then a property of f is that
  $$\Pr[f(\lambda x)] = \lambda^q \Pr[f(x)]$$

N.B. The equation for statistical self-affinity implies that the spectrum of a fractal obeys an inverse q-power law.
Randomisation of the Koch Curve

Aside: all the fractals!

Summary

Objects

Self-similar

Classical

Deterministic

Statistical

Pure

Chaotic

Random

Real

Noise: White to Black

Fourier Spectrum has the property,

\[ f(x) = x^{3/2} \sum_{n=0}^{\infty} 2^{-n^2} \cos(2^n x) \]

<table>
<thead>
<tr>
<th>(k)</th>
<th>Colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>White</td>
</tr>
<tr>
<td>1</td>
<td>Pink</td>
</tr>
<tr>
<td>1.5</td>
<td>Brown</td>
</tr>
<tr>
<td>&gt;2</td>
<td>black</td>
</tr>
</tbody>
</table>
Simulation of Random Fractal Signals

- Basic property is that the Power Spectrum is proportional to $1/k^\delta$
- Basic algorithm is thus:
  1. Compute white noise field $n$.
  2. Compute DFT of $n$ to give $N$ (complex).
  3. Filter $N$ with filter $1/k^{\delta/2}$
  4. Compute inverse DFT of result to give $u[i]$ (real part) - fractal signal.

Noise: White to Black – Elegant Algorithm

1. Compute a random Gaussian distributed array $G_i; i = 0, 1, \ldots, N - 1$ using a conventional Gaussian random number generator, with zero mean and unit variance. Compute a random sequence of uniform distributed numbers $U_i; i = 0, 1, \ldots, N - 1$ in the range zero to one.
2. Calculate the real and imaginary parts: $N_i = G_i \cos 2\pi U_i$ and $M_i = G_i \sin 2\pi U_i$. This defines $G_i$ as the amplitudes and $U_i$ as the phases.
3. Filter $N_i, M_i$ with $W_i = 1/k^{\delta/2}$ to create $N'$ and $M'$. For surfaces $k = \sqrt{k_x^2 + k_y^2}$ and $\beta = 8 - 2D$, and for signals $k = k_x$ and $\beta = 5 - 2D$.
4. Inverse DFT the result using FFT to obtain $n_i = Re \left( F^{-1}(N' + iM') \right)$.

The exponent is $\beta/2$ to ensure that the power spectrum, $P_k$, satisfies

$$P_k = (N'_k)^2 + (M'_k)^2 \propto k^{-\beta}$$
Noise: Non-Fourier

Subdivide – a grid adding noise at each level

\[ d_n = \left( \frac{1}{2} \right)^{nH/2} \]

Matlab example

Perlin Noise – a popular alternative

Similar structure but the sum of weighted frequencies.

Very similar to previous technique but values of ‘d’ from a LUT

Control of Perlin Noise

Parameters control
Tensor Gravity Field Analysis

Aside: Fractional Brownian Motion

\[ B_H(x) = \frac{d^n}{dx^n} n(x) \]

P=0.50, D=1.621

P=0.75, D=1.609
Aside: Fractional Brownian Motion

P=0.10, D=1.690

P=0.25, D=1.641

Adding Deterministic Information

- How can we incorporate \( a \text{ priori} \) information on the large scale structure of a fractal field and thus Tailor it accordingly?

- Replace white noise field \( n \) by \((1-t)n+tp\) where \( p \) is a user defined function and \( t \) (0\(<\)t\(<\)1) is a 'transmission coefficient'.

Example: Sine Wave with Fractal Noise
(t=0.1)

Adding Deterministic Info.

\[ G' + iH' = (1-t)(N + iM) + t(G + iH) \]
Adding Deterministic Info. 

Matlab system from J. Blackledge

Fractal Clouds: $q=1.9$

Fractal Clouds: $q=1.8$

Fractal Clouds: $q=1.7$
Fractal Clouds: $q=1.6$

Fractal Clouds: $q=1.5$

Fractal Clouds: $q=1.4$

Fractal Clouds: $q=1.3$
Novel Methods

- Use of Transparency as a texture modifier for an ellipse

- Use of 3D fractal texture bumpmap
- Fractal transparency for rings on a circle

Divergent fractal fields

- Consider the 2D stochastic fractional divergence equation

\[ \nabla^q \cdot \mathbf{u}(\mathbf{r}) = n(\mathbf{r}), \quad 1 < q < 2 \]
Rotational fractal fields

- Consider the 2D stochastic fractional rotation equation

\[ \nabla^q \times \mathbf{u}(\mathbf{r}) = \mathbf{n}(\mathbf{r}), \quad 1 < q < 2 \]

Fractal flow fields

- Consider the 2D stochastic fractional flow equation

\[ \left( \frac{\partial^{q_1}}{\partial x^{q_1}} + \frac{\partial^{q_2}}{\partial y^{q_2}} \right) u(x, y) = n(x, y), \quad 1 < q_i < 2, \quad i = 1, 2 \]

- \( q_2 < q_1 \): Flow is in y-direction
Controlling a multi-frequency haptic vibration device.

- Random fractal process: PSDF=ck^{-q}
- Ornstein-Uhlenbeck process: ck (k_0^2 + k^2)^{-1}
- Bermann process: c |k|^g (k_0^2 + k^2)^{-1}

Combining the results we can consider the following generalisation:
\[ c |k|^g (k_0^2 + k^2)^{-q} \]

where \( k_0 \) is a 'carrier frequency', c is a constant and \((q, g)\) are fractal parameters.
Language for Fractals

### Lindermayer Systems

Constants | Elements that are fixed
Variables | Elements that can be replaced
Productions | Define how Variables are replaced
Axiom | Specifies how the system begins

---

#### Language for Fractals: Ex 1.

<table>
<thead>
<tr>
<th>Step</th>
<th>L-System Sequence</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$B$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$AB$</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$ABA$</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>$ABAB$</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>$BABABAB$</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>$BBABABABABABAB$</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>$BBABABABABABABABABABAB$</td>
<td>34</td>
</tr>
</tbody>
</table>

**Fib_{j}** = \[
\begin{cases}
1 & j = 1 \\
1 & j = 2 \\
Fib_{j-1} + Fib_{j-2} & j \geq 3
\end{cases}
\]

---

#### Language for Fractals: Ex 2.

<table>
<thead>
<tr>
<th>Variables</th>
<th>$A, B$</th>
</tr>
</thead>
</table>
| Productions | $A \rightarrow AB$ \\
| $B \rightarrow AB$ |
| Axiom     | $A$   |

This has two axioms:

- $A$:

This has two production rules:

- $A \rightarrow AB$:

- $B \rightarrow AB$:
This has two axions:

\[ A : \quad \quad \text{and} \quad \quad B : \]

and two production rules:

\[ A \rightarrow AB : \]
\[ \quad \text{and} \quad \quad B \rightarrow AB : \]

\[ \]

\[ ABAB : \]
Language for Fractals: Ex. 2.

This has two axioms:

\[ A \rightarrow AB \quad \text{and} \quad B \rightarrow AB \]

and two production rules:

\[ A \rightarrow AB \quad \text{and} \quad B \rightarrow AB \]

\[ ABA \quad BABA \quad ABAB \quad ABAB \]

Language for Fractals: FractInt

\[ d \rightarrow \text{A distance defining the unit length.} \]
\[ \delta \rightarrow \text{A positive turning angle specified in degrees.} \]

\[ F \rightarrow \text{Draw a straight line of distance } d. \]

\[ G \rightarrow \text{Move forward, without drawing, distance } d. \]

\[ + \rightarrow \text{Turn counterclockwise } \delta. \]

\[ - \rightarrow \text{Turn clockwise } \delta. \]

\[ \uparrow \downarrow \rightarrow \text{Store current location and heading in the stack.} \]

\[ \downarrow \uparrow \rightarrow \text{Retrieve last location and heading from the stack.} \]

\[ (\times x) \rightarrow \text{Scale the distance value } d \text{ by a factor } x. \]

Language for Fractals: FractInt Ex. 1

Von Koch Curve \{ 
\[ \delta = 60^\circ \]
\[ F \rightarrow -F+FF-F+ \]
\[ \text{Axiom } F \]
\}

Step 1

Step 2

Step 3

Step 4

Language for Fractals: FractInt Ex. 2 – (Revisited)

Dragon \{ 
\[ \delta = 45^\circ \]
\[ F \rightarrow -F+FB-FFA+ \]
\[ A \rightarrow +FB-FFA \]
\[ B \rightarrow -FB+FFA \]
\[ \text{Axiom } FA \]
\}

Step 1

Step 2

Step 3

Step 4

Step 5

Step 10

Step 15

Step 20
Peano:
\[
\delta = 90^\circ
\]
\[F \rightarrow XF\]
\[X \rightarrow XF - XF + XF + XF - XF - XF + XF + XF\]

Axioms:
\[
FX
\]

Step 1
Step 2
Step 3

Plant:
\[
\delta = 10^\circ
\]
\[F \rightarrow \]
\[X \rightarrow Y = F([\times \frac{1}{2}] + \ldots + X) - (\times \frac{1}{2}) F([\times \frac{1}{2}] - \ldots - X)[\times \frac{2}{3}] X\]

Axiom:
\[
++ + + + + + + X
\]
Simulating and Modeling Lichen Growth – Brett Desbenoit, Eric Galin, and Samir Akkouche, LIRIS, CNRS Universite Claude Bernard, Lyon (Poss. wrong one)
Language for Fractals: VR+

Blueberry3D – lots of LOD tricks and billboarding; circa 2003/4

Over 10 years of Fractal curiosity:
Allan Evans, Jonathan Blackledge and all at the old ISS - Institute of Simulation Sciences, De Montfort University
The guys at the Virtual Environment Centre at De Montfort University
All the staff and students at the Manchester Visualization Centre at the University of Manchester
Plus many others (whose credits should be there).

Resources at:
http://wiki.rcs.manchester.ac.uk/community/Fractal_Resources_Tutorial

Research Computing
The University of Manchester
Web: http://www.rcs.manchester.ac.uk/